Paraboson coherent states

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It is known that the defining relations of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ are equivalent to the defining (triple) relations of n pairs of paraboson operators $b_i^\pm$. In particular, the "parabosons of order p" correspond to a unitary irreducible (infinite-dimensional) lowest weight representation $V(p)$ of $\mathfrak{osp}(1|2n)$. Recently we constructed these representations $V(p)$ giving the explicit actions of the $\mathfrak{osp}(1|2n)$ generators. We apply these results for the $n = 2$ case in order to obtain "coherent state" representations of the paraboson operators.

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I. INTRODUCTION

In 1953 Green [1] generalized Fermi-Dirac and Bose-Einstein statistics to parafermi and parabose statistics. Parafermions and parabosons satisfy certain triple relations and the generalization of usual fermion and boson Fock spaces is characterized by a parameter p, the order of the statistics. The parafermion and paraboson Fock spaces of order p can be in principle constructed by means of the so-called Green ansatz [1]. This approach is related to finding a proper basis of an irreducible constituent of a p-fold tensor product [2]. The computational difficulties however are very hard, and did not lead to an explicit solution of the problem. In [3, 4] it was proved that the triple relations for n pairs of parafermions give a set of defining relations for the orthogonal Lie algebra $\mathfrak{so}(2n+1)$. In a similar way it was shown [5] that the triple relations of n pairs of parabosons are defining relations of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [6]. These two important observations imply that the parafermion Fock space of order p is a unitary irreducible representation (unirrep) of $\mathfrak{so}(2n+1)$, more precisely the finite-dimensional unirrep $W(p)$ with lowest weight

$(-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2})$, whereas the paraboson Fock space of order p is the infinite-dimensional unirrep $V(p)$ of $\mathfrak{osp}(1|2n)$ with lowest weight $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$. The construction of these representations, for $n > 1$ and arbitrary p, also turned out to be difficult and was established only quite recently in [7] (for the paraboson Fock spaces) and in [8] (for the parafermion Fock spaces).

In the present paper we apply the results of [7] for the $n = 2$ case in order to obtain "coherent state" representations of two pairs of paraboson operators $b_1^\pm$, $b_2^\pm$. In section II, we describe the paraboson Fock representations $V(p)$ introducing an orthonormal basis and giving the explicit action of the paraboson operators on the basis vectors following [7]. Section III is devoted to the b_1^--coherent states. Common eigenstates of b_1^- and $(b_2^-)^2$ are constructed in section IV. In section V we compute the b_2^--matrix elements thus obtaining coherent state representations of two pairs of paraboson operators.

II. THE PARABOSON FOCK SPACE AND $\mathfrak{osp}(1|2n)$ REPRESENTATIONS

Consider a system of n parabosons $b_j^\pm$ $(j, k, l = 1, 2, \ldots, n;\ \eta, \epsilon, \xi = \pm)$ [1]:

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi. \tag{2.1}$$

The paraboson Fock space $V(p)$ is a Hilbert space with a vacuum $|0\rangle$, defined by means of $\langle 0|0\rangle = 1$, $b_j^-|0\rangle = 0$, $(b_j^\pm)^\dagger = b_j^\mp$, $\{b_j^-, b_k^+\}|0\rangle = p\ \delta_{jk}\,|0\rangle$, where p is a parameter, called the order of the statistics. $V(p)$ was constructed using the relations between n pairs of parabosons and the defining relations of the Lie superalgebra $\mathfrak{osp}(1|2n)$ [6], discovered by Ganchev and Palev [5]. As a basis in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(1|2n)$ consider $h_j = e_{jj} - e_{n+j,n+j}$, $j = 1, \ldots, n$. In terms of the dual basis δ_j of $\mathfrak{h}^*$, the root vectors and the corresponding roots of $\mathfrak{osp}(1|2n)$ are given by:

$$\begin{aligned}
&e_{0,k} - e_{n+k,0} \leftrightarrow -\delta_k, && e_{0,n+k} + e_{k,0} \leftrightarrow \delta_k, \quad k = 1, \ldots, n, \text{ odd},\\
&e_{j,n+k} + e_{k,n+j} \leftrightarrow \delta_j + \delta_k, && e_{n+j,k} + e_{n+k,j} \leftrightarrow -\delta_j - \delta_k,\ j \le k = 1, \ldots, n, \text{ even},\\
&e_{j,k} - e_{n+k,n+j} \leftrightarrow \delta_j - \delta_k, && j \ne k = 1, \ldots, n, \text{ even},
\end{aligned}$$

where e_{ij}, $i, j = 0, 1, \cdots, 2n$ is a matrix with zeros everywhere except a 1 on position (i, j). If we introduce the following multiples of the odd root vectors $b_k^+ = \sqrt{2}(e_{0,n+k} + e_{k,0})$, $b_k^- = \sqrt{2}(e_{0,k} - e_{n+k,0})$, $k = 1, \ldots, n$ the following holds [5]

Theorem 1 *As a Lie superalgebra defined by generators and relations, $\mathfrak{osp}(1|2n)$ is generated by $2n$ odd elements $b_k^\pm$ subject to the relations (2.1).*

Since $\{b_j^-, b_j^+\} = 2h_j, j = 1, \ldots, n$ we have:

Corollary 2 *The paraboson Fock space $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(1|2n)$ with lowest weight $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$.*

Following group theoretical techniques and in particular the chain of subalgebras $\mathfrak{osp}(1|2n) \supset \mathfrak{sp}(2n) \supset \mathfrak{u}(n)$ and the known $\mathfrak{u}(n)$ Gelfand-Zetlin basis (GZ) [9]:

$$|m) \equiv |m)^n \equiv \left| \begin{array}{lllll} m_{1n} & \cdots & \cdots & m_{n-1,n} & m_{nn} \\ m_{1,n-1} & \cdots & \cdots & m_{n-1,n-1} & \\ \vdots & \iddots & & & \\ m_{11} & & & & \end{array} \right) , \tag{2.2}$$

where the top line of the pattern is any partition λ and all m_{ij} satisfy the *betweenness conditions* $m_{i,j+1} \geq m_{ij} \geq m_{i+1,j+1}, 1 \leq i \leq j \leq n-1$, one obtains [7]:

Theorem 3 *If $p > n-1$ an orthonormal basis for the paraboson Fock space $V(p)$ is given by all vectors $|m)$, see* (2.2)*; if $p \in \{1, 2, \ldots, n-1\}$ the basis consists of all vectors $|m)$ of the form* (2.2) *with $m_{p+1,n} = m_{p+2,n} = \ldots = m_{nn} = 0$. The explicit action of the $\mathfrak{osp}(1|2n)$ generators in $V(p)$ is given by*

$$h_k|m) = \left(\frac{p}{2} + \sum_{j=1}^{k} m_{jk} - \sum_{j=1}^{k-1} m_{j,k-1} \right) |m); \tag{2.3}$$

$$\begin{aligned} b_j^+|m) = & \sum_{i_n=1}^{n} \sum_{i_{n-1}=1}^{n-1} \ldots \sum_{i_j=1}^{j} S(i_n, i_{n-1}) \ldots S(i_{j+1}, i_j) \left(\frac{\prod_{k=1}^{j-1}(l_{k,j-1} - l_{i_j,j} - 1)}{\prod_{k\neq i_j=1}^{j}(l_{kj} - l_{i_j,j})} \right)^{1/2} \\ & \times \prod_{r=1}^{n-j} \left(\frac{\prod_{k\neq i_{n-r}=1}^{n-r}(l_{k,n-r} - l_{i_{n-r+1},n-r+1} - 1) \prod_{k\neq i_{n-r+1}=1}^{n-r+1}(l_{k,n-r+1} - l_{i_{n-r},n-r})}{\prod_{k\neq i_{n-r+1}=1}^{n-r+1}(l_{k,n-r+1} - l_{i_{n-r+1},n-r+1}) \prod_{k\neq i_{n-r}=1}^{n-r}(l_{k,n-r} - l_{i_{n-r},n-r} - 1)} \right)^{1/2} \\ & \times F_{i_n}(m_{1n}, m_{2n}, \ldots, m_{nn}) \; |m)_{+i_n,n;+i_{n-1},n-1;\ldots;+i_j,j}; \end{aligned} \tag{2.4}$$

$$b_j^-|m) = \sum_{i_n=1}^{n} \ldots \sum_{i_j=1}^{j} S(i_n, i_{n-1}) S(i_{n-1}, i_{n-2}) \ldots S(i_{j+1}, i_j) \left(\frac{\prod_{k=1}^{j-1} (l_{k,j-1} - l_{i_j,j})}{\prod_{k\neq i_j=1}^{j} (l_{kj} - l_{i_j,j} + 1)} \right)^{1/2}$$

$$\times \prod_{r=1}^{n-j} \left(\frac{\prod_{k\neq i_{n-r}=1}^{n-r} (l_{k,n-r} - l_{i_{n-r+1},n-r+1}) \prod_{k\neq i_{n-r+1}=1}^{n-r+1} (l_{k,n-r+1} - l_{i_{n-r},n-r} + 1)}{\prod_{k\neq i_{n-r+1}=1}^{n-r+1} (l_{k,n-r+1} - l_{i_{n-r+1},n-r+1} + 1) \prod_{k\neq i_{n-r}=1}^{n-r} (l_{k,n-r} - l_{i_{n-r},n-r})} \right)^{1/2}$$

$$\times F_{i_n}(m_{1n}, \ldots, m_{i_n,n} - 1, \ldots, m_{nn})\ |m)_{-i_n,n;-i_{n-1},n-1;\ldots;-i_j,j}; \tag{2.5}$$

$$F_k(m_{1n}, m_{2n}, \ldots, m_{nn}) = (-1)^{m_{k+1,n}+\cdots+m_{nn}} (m_{kn} + n + 1 - k + \mathcal{E}_{m_{kn}}(p-n))^{1/2}$$
$$\times \prod_{j\neq k=1}^{n} \left(\frac{m_{jn} - m_{kn} - j + k}{m_{jn} - m_{kn} - j + k - \mathcal{O}_{m_{jn}-m_{kn}}} \right)^{1/2}. \tag{2.6}$$

Herein $l_{ij} = m_{ij} - i$. An index $\pm i_k, k$ attached as a subscript to $|m)$ indicates a replacement $m_{i_k,k} \to m_{i_k,k} \pm 1$. The function $S(k,l)$ is equal to 1 for $k \leq l$ and -1 for $k > l$. Finally, $\mathcal{E}$ and $\mathcal{O}$ are the even *and* odd *functions defined by $\mathcal{E}_j = 1$ if j is even and 0 otherwise, $\mathcal{O}_j = 1$ if j is odd and 0 otherwise.*

Because of the computational difficulties in the construction of the paraboson Fock spaces the paraboson coherent states (eigenstates of paraboson operators) were constructed only for one pair of paraboson operators [10]. In the rest of the paper we use the results of [7] for the $n = 2$ case in order to obtain "coherent state" representations of two pairs of paraboson operators $b_1^\pm$, $b_2^\pm$.

III. b_1^--COHERENT STATES

First we will construct coherent states of the operator b_1^- as eigenstates in $V(p)$

$$b_1^- \psi = \alpha \psi, \tag{3.1}$$

where α is a complex eigenvalue. Let $|\zeta\rangle \in V(p)$ be a weight vector annihilated by b_1^-, i.e.

$$h_1|\zeta\rangle = \zeta_1|\zeta\rangle, \quad h_2|\zeta\rangle = \zeta_2|\zeta\rangle, \quad b_1^-|\zeta\rangle = 0. \tag{3.2}$$

Lemma 4 *Let $|\zeta\rangle \in V(p)$ be a weight vector annihilated by b_1^- and let $T_1 = b_1^- b_1^+ \in U(\mathfrak{osp}(1|4))$. Then:*

- $$b_1^-(b_1^+)^n|\zeta\rangle = (n + \mathcal{O}_n(2\zeta_1 - 1))(b_1^+)^{n-1}|\zeta\rangle \tag{3.3}$$
- $$T_1(b_1^+)^n|\zeta\rangle = (n + 1 + \mathcal{E}_n(2\zeta_1 - 1))(b_1^+)^n|\zeta\rangle \tag{3.4}$$

For vectors v in $V(p)$ which are T_1-eigenvectors with non-zero eigenvalue, i.e. $T_1 v = \lambda v$, we define $T_1^{-1} v = \lambda^{-1} v$. Then:

- $$T_1^{-1}(b_1^+)^n|\zeta\rangle = (n+1+\mathcal{E}_n(2\zeta_1-1))^{-1}(b_1^+)^n|\zeta\rangle \tag{3.5}$$
- $$(b_1^+T_1^{-1})^n|\zeta\rangle = \prod_{k=1}^{n}(k+\mathcal{O}_k(2\zeta_1-1))^{-1}(b_1^+)^n|\zeta\rangle \tag{3.6}$$

Proof. Equation (3.3) can be proved by induction on n. Formula (3.4) follows directly from (3.3). Because of the diagonal action of T_1 on weight vectors $(b_1^+)^n|\zeta\rangle$ of $V(p)$ and the fact that $n+1+\mathcal{O}_n(2\zeta_1-1) > 0$ one concludes that (3.5) holds. Note that T_1^{-1} is not an element of the enveloping algebra; nevertheless its action on such vectors of $V(p)$ is well defined. The proof of (3.6) uses (3.5) and again induction. □

Formula (3.6) allows us to define a "vertex operator" $\chi(\alpha)$:

$$\chi(\alpha) = \sum_{n=0}^{\infty} \alpha^n (b_1^+T_1^{-1})^n = \frac{1}{1-\alpha b_1^+T_1^{-1}} \tag{3.7}$$

on vectors $|\zeta\rangle$ of the form (3.2). Then we have:

Lemma 5 *Let $|\zeta\rangle \in V(p)$ be a weight vector annihilated by b_1^- that is normalized (i.e. $\langle\zeta|\zeta\rangle = 1$), and $\chi(\alpha)$ a vertex operator of the form (3.7). Then:*

- $\chi(\alpha)|\zeta\rangle \in V(p)$
- *The norm of $\chi(\alpha)|\zeta\rangle$ is given by*

$$\langle\ \chi(\alpha)|\zeta\rangle, \chi(\alpha)|\zeta\rangle\ \rangle = {}_0F_1\left({- \atop \zeta_1}; \left(\frac{\bar{\alpha}\alpha}{2}\right)^2\right) + \frac{\bar{\alpha}\alpha}{2\zeta_1}\, {}_0F_1\left({- \atop \zeta_1+1}; \left(\frac{\bar{\alpha}\alpha}{2}\right)^2\right), \tag{3.8}$$

where ${}_0F_1\left({- \atop a}; x\right)$ is the classical hypergeometric series

$${}_0F_1\left({- \atop a}; x\right) = \sum_{k=0}^{\infty} \frac{x^k}{(a)_k k!}, \quad (a)_k = a(a+1)\cdots(a+k-1) \tag{3.9}$$

- *$\chi(\alpha)|\zeta\rangle$ is an eigenvector of b_1^- with eigenvalue α:*

$$b_1^-\chi(\alpha)|\zeta\rangle = \alpha\chi(\alpha)|\zeta\rangle. \tag{3.10}$$

Proof. The first assertion follows from (3.8), since it is sufficient to show that the norm of the vector is finite. Vectors of different weights are orthogonal, therefore

$$\langle\ \chi(\alpha)|\zeta\rangle, \chi(\alpha)|\zeta\rangle\ \rangle = \sum_{n=0}^{\infty} \bar{\alpha}^n\alpha^n \langle\ (b_1^+T_1^{-1})^n|\zeta\rangle, (b_1^+T_1^{-1})^n|\zeta\rangle\ \rangle.$$

It is not difficult to see that

$$\begin{aligned}
&\langle\, (b_1^+T_1^{-1})^{n+1}|\zeta\rangle, (b_1^+T_1^{-1})^{n+1}|\zeta\rangle\,\rangle\\
&= \langle\, b_1^+T_1^{-1}(b_1^+T_1^{-1})^n|\zeta\rangle, b_1^+T_1^{-1}(b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle\\
&= \langle\, T_1^{-1}(b_1^+T_1^{-1})^n|\zeta\rangle, b_1^-b_1^+T_1^{-1}(b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle\\
&= \langle\, T_1^{-1}(b_1^+T_1^{-1})^n|\zeta\rangle, (b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle\\
&= (n+1+\mathcal{E}_n(2\zeta_1-1))^{-1}\langle\, (b_1^+T_1^{-1})^n|\zeta\rangle, (b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle.
\end{aligned}$$

Now by induction it follows that

$$\langle\, (b_1^+T_1^{-1})^n|\zeta\rangle, (b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle = \prod_{k=1}^{n}(k+\mathcal{O}_k(2\zeta_1-1))^{-1}. \tag{3.11}$$

Therefore

$$\begin{aligned}
&\langle\, \chi(\alpha)|\zeta\rangle, \chi(\alpha)|\zeta\rangle\,\rangle = \sum_{n=0}^{\infty}\bar{\alpha}^n\alpha^n\prod_{k=1}^{n}(k+\mathcal{O}_k(2\zeta_1-1))^{-1}\\
&= 1+\frac{(\frac{\bar{\alpha}\alpha}{2})^2}{(\zeta_1)1!}+\frac{(\frac{\bar{\alpha}\alpha}{2})^4}{(\zeta_1)(\zeta_1+1)2!}+\cdots\\
&+\frac{\bar{\alpha}\alpha}{2\zeta_1}\Big(1+\frac{(\frac{\bar{\alpha}\alpha}{2})^2}{(\zeta_1+1)1!}+\frac{(\frac{\bar{\alpha}\alpha}{2})^4}{(\zeta_1+1)(\zeta_1+2)2!}+\cdots\Big)\\
&= {}_0F_1\left(\begin{matrix}-\\ \zeta_1\end{matrix}; \left(\frac{\bar{\alpha}\alpha}{2}\right)^2\right)+\frac{\bar{\alpha}\alpha}{2\zeta_1}\,{}_0F_1\left(\begin{matrix}-\\ \zeta_1+1\end{matrix}; \left(\frac{\bar{\alpha}\alpha}{2}\right)^2\right).
\end{aligned}$$

Since the classical hypergeometric series (3.9) is convergent for any x one concludes $\chi(\alpha)|\zeta\rangle \in V(p)$.

The last part follows from the following computation:

$$\begin{aligned}
&b_1^-\chi(\alpha)|\zeta\rangle\\
&= b_1^-\big(1+\alpha b_1^+T_1^{-1}+\alpha^2(b_1^+T_1^{-1})(b_1^+T_1^{-1})+\cdots\big)|\zeta\rangle\\
&= \big(b_1^-+\alpha T_1T_1^{-1}+\alpha^2(T_1T_1^{-1})(b_1^+T_1^{-1})+\cdots\big)|\zeta\rangle\\
&= b_1^-|\zeta\rangle+\alpha\big(1+\alpha(b_1^+T_1^{-1})+\alpha^2(b_1^+T_1^{-1})^2+\cdots\big)|\zeta\rangle = \alpha\chi(\alpha)|\zeta\rangle.
\end{aligned}$$

□

Lemma 5 shows that in order to construct b_1^--coherent states we must find a complete basis of the subspace of weight vectors of $V(p)$, annihilated by b_1^-. The weight of the vector $|m)$ is given by $(\frac{p}{2},\frac{p}{2})+(m_{11}, m_{12}+m_{22}-m_{11})$ (see (2.3)). Now if we consider the weights, one could construct vectors

$$|\zeta_{jk}\rangle = \sum_{i=0}^{j} c_i(j,k) \left| \begin{matrix} k+i & j-i \\ j & \end{matrix} \right\rangle, \; k=0,1,\cdots, \quad j=0,1,\cdots,k, \tag{3.12}$$

of weight $(\frac{p}{2},\frac{p}{2})+(j,k)$ with $b_1^-|\zeta_{jk}\rangle = 0$ and $\langle\zeta_{jk}|\zeta_{jk}\rangle = 1$. This construction is given by:

Proposition 6 *An orthonormal basis of the subspace of weight vectors of $V(p)$, annihilated by b_1^- is given by (3.12), where*

$$c_i(j,k) = \sqrt{\binom{k-j+i}{i}} \prod_{r=0}^{k-j} \sqrt{\frac{r+1+\mathcal{O}_r(p-2+2j)}{k+1-r+\mathcal{O}_{k-r}(p-2)}}$$
$$\times \prod_{s=1}^{i} (-1)^{j-s} \sqrt{\frac{(j+1-s+\mathcal{E}_{j-s}(p-2))(k-j+2s+\mathcal{O}_{k+j-1})}{(k+1+s+\mathcal{E}_{k+s-1}(p-2))(k-j+2s-\mathcal{O}_{k+j-1})}}. \tag{3.13}$$

Proof. The action of b_1^- on the GZ basis vectors gives

$$b_1^-|\zeta_{jk}\rangle = \sum_{i=0}^{j-1} \Big(c_{i+1}(j,k)\sqrt{i+1} f_1(k+i, j-i-1) - c_i(j,k)\sqrt{k+i-j+1}$$
$$\times f_2(k+i, j-i-1) \Big) \left| \begin{matrix} k+i & j-i-1 \\ j-1 & \end{matrix} \right\rangle.$$

Therefore

$$c_{i+1}(j,k) = \sqrt{\frac{(k+i-j+1)(j-i+\mathcal{E}_{j-i-1}(p-2))(k-j+2i+2+\mathcal{O}_{k+j-1})}{(i+1)(k+i+2+\mathcal{E}_{k+i}(p-2))(k-j+2i+2-\mathcal{O}_{k+j-1})}}$$
$$\times (-1)^{j-i-1} c_i(j,k).$$

Clearly, the coefficients $c_i(j,k)$ (see (3.13)) satisfy the last equation. The norm condition $\sum_{i=0}^{j} c_i(j,k)^2 = 1$ is equivalent to the following identity:

$$\sum_{i=0}^{j} \binom{k-j+i}{i} \prod_{r=1}^{i} \frac{(j+1-r+\mathcal{E}_{j-r}(p-2))(k-j+2r+\mathcal{O}_{k+j-1})}{(k+1+r+\mathcal{E}_{k+r-1}(p-2))(k-j+2r-\mathcal{O}_{k+j-1})}$$
$$= \prod_{r=0}^{k-j} \frac{(k+1-r+\mathcal{O}_{k-r}(p-2))}{(r+1+\mathcal{O}_r(p-2+2j))}, \tag{3.14}$$

which can be proved using hypergeometric summations.

In order to show that vectors of the form (3.12) form a basis of the space annihilated by b_1^-, one uses a weight argument and the explicit action of b_1^-, given by (2.5). Note that in

$V(p)$, the multiplicity of the weight $(\frac{p}{2}, \frac{p}{2}) + (j,k)$ is given by $\min(j+1, k+1)$. For $k=0$, it follows from (2.5) that there is only one vector annihilated by b_1^-. For $k=1$, (2.5) and the above multiplicity allow the construction of only two vectors annihilated by b_1^-. More generally, the multiplicity argument and (2.5) yield at most $k+1$ vectors annihilated by b_1^- for a given k-value. Since all vectors (3.12) are linearly independent, the statement follows. $\square$

Combination of Proposition 6 and the previous lemma now yields the following result:

Proposition 7 *A complete set of b_1^--coherent states $b_1^-\psi_{jk}(\alpha) = \alpha\psi_{jk}(\alpha)$ is defined by*

$$\psi_{jk}(\alpha) = \chi(\alpha)|\zeta_{jk}\rangle, \quad k = 0, 1, \cdots; \; j = 0, 1, \cdots, k, \tag{3.15}$$

where $\chi(\alpha)$ and $|\zeta_{jk}\rangle$ are given by (3.7) and (3.12)-(3.13) resp. and

$$\langle\, \psi_{jk}(\alpha), \psi_{jk}(\alpha)\rangle = {}_0F_1\left({- \atop \frac{p}{2}+j} ; \left(\frac{\bar{\alpha}\alpha}{2}\right)^2 \right) + \frac{\bar{\alpha}\alpha}{p+2j}\,{}_0F_1\left({- \atop \frac{p}{2}+j+1} ; \left(\frac{\bar{\alpha}\alpha}{2}\right)^2 \right) \tag{3.16}$$

Proof. The only part left to be proved is (3.16). It follows directly from the fact that $T_1|\zeta_{jk}\rangle = 2h_1|\zeta_{jk}\rangle = (p+2j)|\zeta_{jk}\rangle$ and (3.8). $\square$

IV. b_1^-- AND $(b_2^-)^2$-COHERENT STATES

In the previous section we have so far constructed only the b_1^--coherent states. Since we are dealing with two pairs of parabosons $b_1^\pm, b_2^\pm$, the question is if one can construct "bicoherent states". The problem however, is that b_1^- and b_2^- do not commute, so they cannot have common eigenstates. However, b_1^- and $(b_2^-)^2$ commute. In this section we construct common eigenstates of the operators b_1^- and $(b_2^-)^2$. Using the defining triple paraboson relations (2.1) it is straightforward to see that the operator $(b_2^-)^2$ commutes with b_1^-. Hence, the action of $(b_2^-)^2$ also commutes with T_1 and T_1^{-1}. Therefore one concludes that $(b_2^-)^2$ commutes with $\chi(\alpha)$:

$$(b_2^-)^2\psi_{jk}(\alpha) = \chi(\alpha)(b_2^-)^2|\zeta_{jk}\rangle. \tag{4.1}$$

Note that $(b_2^-)^2|\zeta_{jk}\rangle$ is a vector of weight $(\frac{p}{2}, \frac{p}{2}) + (j, k-2)$ and that $b_1^-(b_2^-)^2|\zeta_{jk}\rangle = (b_2^-)^2 b_1^-|\zeta_{jk}\rangle = 0$. Because of the fact that there is only one vector of weight $(\frac{p}{2}+j, \frac{p}{2}+k-2)$ annihilated by b_1^- one concludes that $(b_2^-)^2|\zeta_{jk}\rangle = c|\zeta_{j,k-2}\rangle$. We could find the constant c by computing $(b_2^-)^2|\zeta_{jk}\rangle$ on one of the GZ basis vectors of $|\zeta_{jk}\rangle$ and compare the result with the same GZ vector in $|\zeta_{j,k-2}\rangle$. The result is as follows

$$(b_2^-)^2|\zeta_{jk}\rangle = \sqrt{(k-1-j+\mathcal{E}_{k-j})(p+k-2+j+\mathcal{O}_{k+j})}|\zeta_{j,k-2}\rangle. \tag{4.2}$$

Therefore

$$(b_2^-)^2\psi_{jk}(\alpha) = \sqrt{(k-1-j+\mathcal{E}_{k-j})(p+k-2+j+\mathcal{O}_{k+j})}\psi_{j,k-2}(\alpha). \tag{4.3}$$

Now it is not difficult to construct common b_1^-- and $(b_2^-)^2$-coherent states

$$b_1^-\Psi_{jl}(\alpha,\beta) = \alpha\Psi_{jl}(\alpha,\beta), \quad (b_2^-)^2\Psi_{jl}(\alpha,\beta) = \beta\Psi_{jl}(\alpha,\beta), \tag{4.4}$$

where

$$\Psi_{jl}(\alpha,\beta) = \sum_{k=0}^{\infty} \frac{\beta^{k+\lfloor\frac{l}{2}\rfloor}}{\sqrt{(2k)!!(p+2l)(p+2l+2)\cdots(p+2l+2(k-1))}}\psi_{j,2k+l}(\alpha), \tag{4.5}$$
$$j=0,1,\ldots; \quad l=j,j+1.$$

The index jl in Ψ_{jl} refers to the weight of the lowest weight vector in the expansion of Ψ_{jl} in the GZ-basis (just as this was the case for ψ_{jk}). The states ψ_{jk} can be considered as bicoherent states.

V. b_2^--MATRIX ELEMENTS

As we mentioned earlier, b_2^- does not commute with b_1^-, and so the b_1^--coherent states $\psi_{jk}(\alpha)$ constructed in section III are no eigenvectors of b_2^-. It turns out however, that the matrix elements of b_2^- acting on these coherent states $\psi_{jk}(\alpha)$ can still be computed explicitly. This is performed in the current section. Let us consider the operator $\chi(\alpha)$ acting on a weight vector $|\zeta\rangle$ annihilated by b_1^-, and apply formula (3.6). Then one could write

$$\begin{aligned}
\chi(\alpha)|\zeta\rangle &= \sum_{n=0}^{\infty}\alpha^n(b_1^+T_1^{-1})^n|\zeta\rangle \\
&= \sum_{n=0}^{\infty}\alpha^{2n}(b_1^+T_1^{-1})^{2n}|\zeta\rangle + \sum_{n=0}^{\infty}\alpha^{2n+1}(b_1^+T_1^{-1})^{2n+1}|\zeta\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!(\zeta_1)_n}\big(\frac{\alpha b_1^+}{2}\big)^{2n}|\zeta\rangle + \frac{\alpha b_1^+}{2\zeta_1}\sum_{n=0}^{\infty}\frac{1}{n!(\zeta_1+1)_n}\big(\frac{\alpha b_1^+}{2}\big)^{2n}|\zeta\rangle \\
&= {}_0F_1\left(\begin{matrix}-\\ \zeta_1\end{matrix};\left(\frac{\alpha b_1^+}{2}\right)^2\right)|\zeta\rangle + \frac{\alpha b_1^+}{2\zeta_1}\,{}_0F_1\left(\begin{matrix}-\\ \zeta_1+1\end{matrix};\left(\frac{\alpha b_1^+}{2}\right)^2\right)|\zeta\rangle.
\end{aligned} \tag{5.1}$$

Note that, by a weight argument, $\langle\psi_{j',k'}(\alpha')|b_2^-|\psi_{jk}(\alpha)\rangle$ can be nonzero only if $k'=k-1$. First, use (5.1) and the fact that b_2^- commutes with $(b_1^+)^2$:

$$\begin{aligned}
&\langle\psi_{j',k-1}(\alpha')|b_2^-|\psi_{jk}(\alpha)\rangle\\
&=\langle\psi_{j',k-1}(\alpha')|b_2^-\left({}_0F_1\left({- \atop \frac{p}{2}+j};\left(\frac{\alpha b_1^+}{2}\right)^2\right)+\frac{\alpha b_1^+}{p+2j}\,{}_0F_1\left({- \atop \frac{p}{2}+j+1};\left(\frac{\alpha b_1^+}{2}\right)^2\right)\right)|\zeta_{jk}\rangle\\
&=\langle\psi_{j',k-1}(\alpha')|\,{}_0F_1\left({- \atop \frac{p}{2}+j};\left(\frac{\alpha b_1^+}{2}\right)^2\right)b_2^-|\zeta_{jk}\rangle\\
&+\langle\psi_{j',k-1}(\alpha')|\,{}_0F_1\left({- \atop \frac{p}{2}+j+1};\left(\frac{\alpha b_1^+}{2}\right)^2\right)\frac{\alpha b_2^- b_1^+}{p+2j}|\zeta_{jk}\rangle.
\end{aligned}$$

Now, use the action of b_1^+ to the left and the action $b_1^-\psi_{j,k-1}(\alpha')=\alpha'\psi_{j,k-1}(\alpha')$. This yields:

$$\begin{aligned}
&\langle\psi_{j',k-1}(\alpha')|b_2^-|\psi_{jk}(\alpha)\rangle\\
&={}_0F_1\left({- \atop \frac{p}{2}+j};\left(\frac{\alpha\bar{\alpha}'}{2}\right)^2\right)\langle\psi_{j',k-1}(\alpha')|b_2^-|\zeta_{jk}\rangle\\
&+{}_0F_1\left({- \atop \frac{p}{2}+j+1};\left(\frac{\alpha\bar{\alpha}'}{2}\right)^2\right)\frac{\alpha}{p+2j}\langle\psi_{j',k-1}(\alpha')|b_2^-b_1^+|\zeta_{jk}\rangle.
\end{aligned}$$

Therefore the computation is reduced to computing the above two matrix elements. Using the explicit form of $|\zeta_{jk}\rangle$, the action of b_1^+ and b_2^- on GZ-basis vectors, and the expansion of $\psi_{j',k-1}(\alpha')$ in terms of GZ-basis vectors one finds:

$$\langle\psi_{j',k-1}(\alpha')|b_2^-|\zeta_{jk}\rangle=\begin{cases}\dfrac{p-2}{p-2+2j}\sqrt{k-j+\mathcal{O}_{k-j}(p-1+2j)} & \text{if } j'=j\\[2ex] 2(-1)^{j-1}\bar{\alpha}'\dfrac{\sqrt{j(p-2+j)(p+k+j-1-\mathcal{E}_{k-j})}}{(p+2j-2)^{3/2}} & \text{if } j'=j-1\\[2ex] 0 & \text{otherwise}\end{cases}$$

$$\langle\psi_{j',k-1}(\alpha')|b_2^-b_1^+|\zeta_{jk}\rangle=\begin{cases}-\bar{\alpha}'\dfrac{p-2}{p+2j}\sqrt{k-j+\mathcal{O}_{k-j}(p-1+2j)} & \text{if } j'=j\\[2ex] 2(-1)^{j}\sqrt{\dfrac{(j+1)(p-1+j)(k-j-\mathcal{O}_{k-j})}{(p+2j)}} & \text{if } j'=j+1\\[2ex] 0 & \text{otherwise}\end{cases}$$

Hence $\langle\psi_{j',k-1}(\alpha')|b_2^-|\psi_{jk}(\alpha)\rangle$ is 0 for $j'\neq j-1, j, j+1$. For the other three cases it follows from the above formulas. For example if $j'=j-1$ it is given by:

$$\langle\psi_{j-1,k-1}(\alpha')|b_2^-|\psi_{jk}(\alpha)\rangle$$
$$= {}_0F_1\left(\begin{array}{c} - \\ \frac{p}{2}+j \end{array}; \left(\frac{\alpha\bar{\alpha}'}{2}\right)^2\right) 2(-1)^{j-1}\bar{\alpha}'\frac{\sqrt{j(p-2+j)(p+k+j-1-\mathcal{E}_{k-j})}}{(p+2j-2)^{3/2}}.$$

VI. CONCLUSION

We constructed coherent state representations of the $\mathfrak{osp}(1|4)$ superalgebra generated by two pairs of paraboson operators $b_1^\pm$, $b_2^\pm$. The interesting and important problem of the decomposition of unity will be given in a future publication. We also hope to generalize the present results to the n-mode paraboson superalgebra $\mathfrak{osp}(1|2n)$.

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